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Characterization of Generalized Convex Functions by Their Best Approximation in Sign-Monotone Norms

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Let $\{u_0, u_1, \ldots u_{n-1}\}$ and $\{u_0, u_1, \ldots, u_n\}$ be Tchebycheff-systems of continuous functions on $[a, b]$ and let $f \in C[a, b]$ be generalized convex with respect to $\{u_0, u_1, \ldots, u_{n-1}\}$. In a series of papers ([1], [2], [3]) D. Amir and Z. Ziegler discuss some properties of elements of best approximation to f from the linear spans of $\{u_0, u_1, \ldots, u_{n-1}\}$ and $\{u_0, u_1, \ldots, u_n\}$ in the L_p-norms, $1 \leqslant p \leqslant \infty$, and show (under different conditions for different values of p) that these properties, when valid for all subintervals of $[a, b]$, can characterize generalized convex functions. Their methods of proof rely on characterizations of elements of best approximation in the L_p-norms, specific for each value of p. This work extends the above results to approximation in a wider class of norms, called "sign-monotone," [6], which can be defined by the property: $|f(x)| \leqslant |g(x)|, f(x)g(x) \geqslant 0, a \leqslant x \leqslant b$, imply $\|f\| \leqslant \|g\|$. For sign-monotone norms in general, there is neither uniqueness of an element of best approximation, nor theorems characterizing it. Nevertheless, it is possible to derive many common properties of best approximants to generalized convex functions in these norms, by means of the necessary condition proved in [6]. For $\{u_0, u_1, \ldots, u_n\}$ an Extended-Complete Tchebycheff-system and $f \in C^{(n)}[a, b]$ it is shown that the validity of any of these properties on all subintervals of $[a, b]$, implies that f is generalized convex. In the special case of f monotone with respect to a positive function $u_0(x)$, a converse theorem is proved under less restrictive assumptions.

1. NOTATIONS AND PRELIMINARIES

Let $\{u_0, u_1, \ldots, u_{n-1}\}$ and $\{u_0, u_1, \ldots, u_{n-1}, u_n\}$ be positive Tchebycheff-systems (T-systems) on $[a, b]$, i.e.,

$$\Delta\begin{pmatrix} u_0, u_1, \ldots, u_k \\ x_0, x_1, \ldots, x_k \end{pmatrix} = \begin{vmatrix} u_0(x_0) & u_0(x_1) & \cdots & u_0(x_k) \\ u_1(x_0) & u_1(x_1) & \cdots & u_1(x_k) \\ \vdots & & & \\ u_k(x_0) & u_k(x_1) & \cdots & u_k(x_k) \end{vmatrix} > 0 \tag{1.1}$$

for all $a \leqslant x_0 < x_1 < \cdots < x_k \leqslant b$ and $k = n - 1$ or n. We denote by $\Lambda_{n-1} = \Lambda_{n-1}[a, b]$ ($\Lambda_n = \Lambda_n[a, b]$) the linear span of $\{u_0, u_1, \ldots, u_{n-1}\}$

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($\{u_0, u_1, \ldots, u_n\}$) and by $C(u_0, u_1, \ldots, u_{n-1})$ the cone of all "generalized convex" functions, i.e., functions for which

$$\Delta\begin{pmatrix} u_0, u_1, \ldots, u_{n-1}, f \\ x_0, x_1, \ldots, x_{n-1}, x_n \end{pmatrix} = \Delta(f, x_0, x_1, \ldots, x_n) = \begin{vmatrix} u_0(x_0) & \cdots & u_0(x_n) \\ u_1(x_0) & \cdots & u_1(x_n) \\ \vdots & & \\ u_{n-1}(x_0) & \cdots & u_{n-1}(x_n) \\ f(x_0) & \cdots & f(x_n) \end{vmatrix} \geqslant 0. \quad (1.2)$$

As introduced in [6] we call a norm $\|\cdot\|$ defined on $C[a, b]$ "sign-monotone" provided:

$$f(x)\cdot g(x) \geqslant 0, \quad |f(x)| \leqslant |g(x)|, \quad a \leqslant x \leqslant b, \quad \text{implies} \quad \|f\| \leqslant \|g\|. \quad (1.3)$$

As shown in [6] this class of norms is wider than the class of "monotone norms" (norms for which $|f(x)| \leqslant |g(x)|$, $a \leqslant x \leqslant b$, implies $\|f\| \leqslant \|g\|$). All weighted L_p-norms (denoted by $\|\cdot\|_p$), $1 \leqslant p \leqslant \infty$, are obviously monotone norms (and thus sign-monotone norms).

Properties of monotone norms and sign-monotone norms are discussed in detail in [5] and [6]. One of these properties which will be used extensively in the sequel is

$$\|f\| \leqslant 2\|1\|\cdot\|f\|_\infty \qquad f \in C[a, b] \quad (1.4)$$

where $\|f\|_\infty = \max_{a\leqslant x\leqslant b} |f(x)|$.

Following the notation in [4] we call an isolated zero x_0 of $f \in C[a, b]$ in (a, b) "nonnodal" if f does not change sign at x_0. All other isolated zeroes of f are called "nodal." The number of isolated zeroes of f in $[a, b]$ is denoted by $z(f)$ while their number with nonnodal zeroes counted twice is denoted by $\tilde{z}(f)$.

In [6] a necessary condition for $T_{n-1}^* \in \Lambda_{n-1}$ to be a "polynomial" of best approximation (p.b.a) to a function $f \in C[a, b]$ with respect to a sign-monotone norm is given. This condition is valid under any one of the following restrictions imposed on the T-system $\{u_0, u_1, \ldots, u_{n-1}\}$:

$$\{u_0, u_1, \ldots, u_{n-2}\} \text{ is a } T\text{-system on } [a, b] \quad (1.5)$$

$$\{u_0, u_1, \ldots, u_{n-1}\} \text{ is a } T\text{-system on } [a'b'], \qquad a' < a < b < b'. \quad (1.6)$$

or

$$\{u_0, u_1, \ldots, u_{n-1}\} \text{ is extended of order 2 on } [a, b]. \quad (1.7)$$

THEOREM 1.1 [6]. *Let $f \in C[a, b]$ and let $\{u_0, u_1, \ldots, u_{n-1}\}$ be a T-system satisfying one of the properties* (1.5), (1.6) *or* (1.7). *If T_{n-1}^* is a p.b.a. to f from Λ_{n-1}^* in a sign-monotone norm, such that $f - T_{n-1}^*$ has only isolated zeroes, then $\tilde{z}(f - T_{n-1}) \geqslant n$.*

This result is best possible in the sense that there are monotone norms for which $\tilde{z}(f - T_{n-1}^*) = n > z(f - T_{n-1}^*)$. (See ex. 2.1 in [5]).

There are also monotone norms for which T_{n-1}^* is nonunique [5], [6] and therefore we refer hereby to "an" element of best approximation.

In this work we also discuss the problem of best approximating a given function $f \in [a, b]$ on subintervals of $[a, b]$. When the norm under consideration is an L_p-norm $(1 \leqslant p \leqslant \infty)$, one naturally uses the same norm on any subinterval of $[a, b]$. It is of special interest to see that with each sign-monotone norm on $C[a, b]$, it is possible to associate sign-monotone norms $\|\cdot\|_I$ defined on $C(I)$, $I = [\alpha, \beta] \subset [a, b]$, in such a way that $T(f, \|\cdot\|_I)$, a p.b.a. to $f \in C[a, b]$ on I from $\Lambda_n(I)$, with respect to $\|\cdot\|_I$, is a "continuous" function of the interval. Since $T(f, \|\cdot\|_I)$ may be nonunique, the concept of "continuity" here is explained in detail in the following theorem:

THEOREM 1.2. *For every sign-monotone norm $\|\cdot\|$ defined on $C[a, b]$ it is possible to define sign-monotone norms $\|\cdot\|_I$ on $C(I)$ for all subintervals I of $[a, b]$ in such a way that if $I_m = [\alpha_m, \beta_m] \subset [a, b]$, $m = 1, 2, 3, \ldots$ satisfy $\lim_{m\to\infty} \alpha_m = \alpha_0 < \beta_0 = \lim_{m\to\infty} \beta_m$, $[\alpha_0, \beta_0] = I_0 \subset [a, b]$,* and *if $T_m = T(f, \|\cdot\|_{I_m})$ is a p.b.a. to $f \in C[a, b]$ from Λ_n on I_m with respect to the norm $\|\cdot\|_{I_m}$, then there exists a subsequence $\{T_{m_j}\}_{j=1}^{\infty}$ converging uniformly on $[a, b]$ to a polynomial $T_0 = T(f, \|\cdot\|_{I_0}) \in \Lambda_n$ which is a p.b.a. to f from Λ_n on I_0 with respect to the norm $\|\cdot\|_{I_0}$.*

Proof. Let us associate with each subinterval $I = [\alpha, \beta] \subset [a, b]$ the linear function $\phi_I(t) = [(\alpha - \beta)/(a - b)](t - b) + \beta$ which maps $[a, b]$ onto $[\alpha, \beta]$, and for a given sign-monotone norm $\|\cdot\|$ on $C[a, b]$, define

$$\| g \|_I = \| g(\phi_I)\| \qquad g \in C(I). \tag{1.8}$$

Since $g(\phi_I) \in C[a, b]$, $\|\cdot\|_I$ is well defined, and moreover it is easy to see that $\|\cdot\|_I$ is a sign-monotone norm on $C(I)$. Therefore (1.4) implies that for every $g \in C(I)$

$$\| g \|_I \leqslant 2 \| 1 \|_I \max_{x\in I} | g(x)|.$$

But from the definition (1.8) it follows that $\| 1 \|_I = \| 1 \|$ and thus, if $f \in C[a, b]$ we get

$$\| f \|_I \leqslant 2 \| 1 \| \| f \|_\infty \qquad (\| f \|_\infty = \max_{a\leqslant x\leqslant b} | f(x)|) \tag{1.9}$$

where by $\|f\|_I$ we mean the norm of the restriction of f to the subinterval I.
Under the assumptions of the theorem it is also clear that

$$\lim_{m\to\infty} \|f\|_{I_m} = \|f\|_{I_0} \qquad f \in C[a, b]. \tag{1.10}$$

This follows easily from (1.9) and from the fact that the sequence of functions $\{f_m\}_{m=1}^{\infty}$, $f_m(x) = f(\phi_{I_m}(x))$ $m = 1, 2,\ldots,$ converges uniformly on $[a, b]$ to $f_0(x) = f(\phi_{I_0}(x)) \in C[a, b]$.

Now, using the following notations: $E_m = \|f - T_m\|_{I_m}$, $m = 1, 2,\ldots,$ $E_0 = \inf_{T\in\Lambda_n} \|f - T\|_I$. and $E = \inf_{T\in\Lambda_{n-1}} \|f - T\|_\infty = \|f - \hat{T}\|_\infty$, we first show that there exists a constant M independent of m such that

$$\| T_m \|_{I_m} \leqslant M, \qquad m = 1, 2,\ldots . \tag{1.11}$$

Indeed, by (1.9):

$$\begin{aligned} \| T_m \|_{I_m} &\leqslant \| T_m - f\|_{I_m} + \|f\|_{I_m} \\ &\leqslant \|f - \hat{T}\|_{I_m} + \|f\|_{I_m} \leqslant 2 \| 1 \| E + 2 \| 1 \| \cdot \|f\|_\infty . \end{aligned}$$

Standard compactness arguments (see for example [7] p. 16) imply that $\{T_m\}$ has a subsequence, denoted again by $\{T_m\}$, converging uniformly to $T_0 \in \Lambda_{n-1}$. But (1.8), (1.4) and the triangle inequality yield:

$$\begin{aligned} | E_m - \|f - T_0\|_{I_0} | &= | \|f - T_m\|_{I_m} - \|f - T_0\|_{I_0} | \\ &= | \|f(\phi_{I_m}) - T_m(\phi_{I_m})\| - \|f(\phi_{I_0}) - T_0(\phi_{I_0})\| | \\ &\leqslant \|f(\phi_{I_m}) - f(\phi_{I_0}) + T_0(\phi_{I_0}) - T_m(\phi_{I_m})\| \\ &\leqslant \|f(\phi_{I_m}) - f(\phi_{I_0})\| + \| T_0(\phi_{I_0}) - T_m(\phi_{I_m})\| \\ &\leqslant 2 \| 1 \| \|f(\phi_{I_m}) - f(\phi_{I_0})\|_\infty \\ &\quad + 2 \| 1 \| \| T_0(\phi_{I_0}) - T_m(\phi_{I_m})\|_\infty . \end{aligned}$$

Since the last expression tends to zero as $m \to \infty$, we get

$$\lim_{m\to\infty} E_m = \|f - T_0\|_{I_0} . \tag{1.12}$$

On the other hand, if $\tilde{T}$ is any p.b.a to f on I_0 in the norm $\|\cdot\|_{I_0}$ then:

$$\begin{aligned} E_m &= \|f - T_m\|_{I_m} \leqslant \|f - \tilde{T}\|_{I_m} = \|f(\phi_{I_m}) - \tilde{T}(\phi_{I_m})\| \\ &\leqslant \|f(\phi_{I_m}) - f(\phi_{I_0})\| + \|f(\phi_{I_0}) - \tilde{T}(\phi_{I_0})\| + \| \tilde{T}(\phi_{I_0}) - \tilde{T}(\phi_{I_m})\| \\ &\leqslant 2 \| 1 \| \|f(\phi_{I_m}) - f(\phi_{I_0})\|_\infty + \|f - \tilde{T}\|_{I_0} + 2 \| 1 \| \| \tilde{T}(\phi_{I_0}) - \tilde{T}(\phi_{I_m})\|_\infty \end{aligned}$$

and as $m \to \infty$ we get

$$\lim_{m\to\infty} E_m \leqslant \| f - \tilde{T} \|_{I_0} = E_0 .$$

This, combined with (1.12) completes the proof of the theorem.

For the sup-norm on $[a, b]$ definition (1.8) yields the sup-norm on every subinterval of $[a, b]$, and for the L_p-norms, $1 \leqslant p < \infty$, it yields $\| g \|_I = (\int_a^b | g(\phi_I(t))|^p \, dt)^{1/p}$ which is a constant multiple of $(\int_\alpha^\beta | g(x)|^p \, dx)^{1/p}$. Yet, it should be noted that for our purposes, any definition having the continuity property expressed in Theorem 1.2 is applicable.

Therefore we call a class of sign-monotone norms:

$$\{\| \cdot \|_I , \| \cdot \|_I \text{ is defined on } C(I), I \subset [a, b]\}$$

"continuous", provided there is continuity of the polynomials of best approximation to any $f \in C[a, b]$ with respect to the change of the subintervals, in the sense of Theorem 1.2.

It is interesting to note that the definition of $\| \cdot \|_I$ in [5] as

$$\| f \|_I = \sup\{\| h \|, h \in C[a, b] \mid h(x)| \leqslant | f(x)| \; a \leqslant x \leqslant b, \quad h(x) = 0 \; x \in [a, b] - I\}$$

which is natural for the problem discussed there, does not always yield a continuous class of norms.

2. Necessary Conditions for Generalized Convexity

In this section we study some properties of polynomials of best approximation from Λ_{n-1} and Λ_n in a given sign-monotone norm $\| \cdot \|$ to functions $f \in C[a, b]$ which satisfy either

$$f \in C(u_0 , u_1 ,..., u_{n-1}) \tag{2.1}$$

or

$$f \in C(u_0 , u_1 ,..., u_{n-1}) - \Lambda_{n-1} \qquad \text{for all} \quad [\alpha, \beta] \subset [a, b]. \tag{2.2}$$

It is easily seen that (2.2) implies

$$\Delta(f, x_0 , x_1 ,..., x_n) > 0 \qquad \text{for all} \quad a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b \tag{2.3}$$

and therefore $\{u_0 , u_1 ,..., u_{n-1} , f\}$ form a T-system in this case.

The first two lemmas concern functions which satisfy (2.1).

LEMMA 2.1. *Let $f \in C[a, b]$ satisfy* (2.1). *Then either f has only isolated zeroes with $0 \leqslant \tilde{z}(f) \leqslant n$, or it vanishes on a subinterval $[\alpha, \beta] \subset [a, b]$ and is different from zero elsewhere in $[a, b]$.*

Proof. Suppose first that f vanishes on a subinterval $[\alpha, \beta] \subset [a, b]$ and that it has a zero $\gamma \notin [\alpha, \beta]$. Without loss of generality we treat only the case $\beta < \gamma \leqslant b$. For $\{t_0, t_1, \ldots, t_{n-1}\} \subset [\alpha, \beta]$ and x a point where $f(x) \neq 0$ $\beta < x < \gamma$ we then obviously have:

$$\Delta(f, t_0, t_1, \ldots, t_{n-1}, x) \cdot \Delta(f, t_0, t_1, \ldots, t_{n-2}, x, \gamma) < 0$$

which contradicts (2.1).

In case f has only isolated zeroes and $z(f) \geqslant n + 1$ the same arguments as above lead to a contradiction. Suppose therefore that $\tilde{z}(f) > n$ and f has less than $n + 1$ isolated zeroes (and thus at least one of them nonnodal). We construct a set of $n + 1$ points $a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b$ such that the values $\{f(t_i)\}_{i=0}^{n}$ alternate in sign in the sense that

$$(-1)^{i+1} f(t_{n-i}) \geqslant 0 \qquad i = 0, 1, \ldots, n \tag{2.4}$$

with at least one of them nonzero. This can be done by including in $\{t_0, t_1, \ldots, t_n\}$ all the zeroes of f, and if t_ν is a nonnodal zero of f then either $t_{\nu-1} = t_\nu - \epsilon$ or $t_{\nu+1} = t_\nu + \epsilon$ ($\epsilon > 0$ small enough). In view of (2.4), the determinant $\Delta(f, t_0, t_1, \ldots, t_n)$ expanded along its last row is easily seen to be negative, in contradiction to (2.1).

LEMMA 2.2. *Let $f \in C[a, b]$ satisfy* (2.1). *If $f(x)$ vanishes on a subinterval $[\alpha, \beta] \subset [a, b]$ then if $\beta < b$*

$$f(x) > 0, \qquad \beta < x \leqslant b \tag{2.5}$$

and if $\alpha > a$

$$(-1)^n f(x) > 0, \qquad a \leqslant x < \alpha. \tag{2.6}$$

If $\tilde{z}(f) = n$ and $a \leqslant z_1 < z_2 < \cdots < z_k \leqslant b$ are the zeroes of $f(x)$ in $[a, b]$ then for $z_k < b$

$$f(x) > 0, \qquad z_k < x \leqslant b \tag{2.7}$$

while for $z_k = b$

$$f(x) < 0, \qquad z_{k-1} < x < b. \tag{2.8}$$

Proof. In case $f(x)$ vanishes on $[\alpha, \beta]$, $\beta < b$ we have for $\alpha \leqslant t_0 < t_1 < \cdots < t_{n-1} < \beta < x \leqslant b$:

$$\Delta(f, t_0, t_1, ..., t_{n-1}, x) = f(x) \cdot \Delta \begin{pmatrix} u_0, u_1, ..., u_{n-1} \\ t_0, t_1, ..., t_{n-1} \end{pmatrix} \geqslant 0.$$

This proves (2.5) since

$$\Delta \begin{pmatrix} u_0, u_1, ..., u_{n-1} \\ t_0, t_1, ..., t_{n-1} \end{pmatrix} > 0$$

and by Lemma 2.1 $f(x) \neq 0$. For $\alpha > a$ (2.6) is proved similarly. If $\tilde{z}(f) = n$ and $z_k < x \leqslant b$, we construct a set of points $a \leqslant t_0 < t_1 < \cdots < t_{n-1} < x \leqslant b$ for which $(-1)^{i+1} f(t_{n-i}) \geqslant 0$ $i = 1, 2, ..., n$. This can be done by including in $\{t_0, t_1, ..., t_{n-1}\}$ all the zeroes of $f(x)$ where for each nonnodal zero t_ν, either $t_{\nu-1} = t_\nu - \epsilon$ or $t_{\nu+1} = t_\nu + \epsilon$ ($\epsilon > 0$ small enough). Thus

$$\Delta(f, t_0, t_1, ..., t_{n-1}, x) = f(x) \cdot \Delta \begin{pmatrix} u_0, u_1, ..., u_{n-1} \\ t_0, t_1, ..., t_{n-1} \end{pmatrix} + \text{nonpositive terms}$$

and (2.7) is proved as above. To prove (2.8) we calculate $\Delta(f, t_0, t_1, ..., t_{n-2}, x, b)$ where $\{t_0, t_1, ..., t_{n-2}\}$ is chosen in a similar manner.

In the following theorems which are easy consequences of Theorem 1.1 and the lemmas, we assume that $\{u_0, u_1, ..., u_{n-1}\}$ is a T-system satisfying one of the conditions (1.5), (1.6) or (1.7).

THEOREM 2.1. *Let $f \in C[a, b]$ satisfy* (2.1), *and let $T_{n-1}^* \in \Lambda_{n-1}$ be any p.b.a. to f from Λ_{n-1} in a sign-monotone norm $\|\cdot\|$. Then either all the zeroes of $f - T_{n-1}^*$ are isolated with $\tilde{z}(f - T_{n-1}^*) = n$ or $f - T_{n-1}^*$ vanishes on a subinterval of* $[a, b]$ *and is different from zero elsewhere in* $[a, b]$. *In both cases, if $f(b) - T_{n-1}^*(b) \neq 0$ then the last sign of $f - T_{n-1}^*$ in $[a, b]$ is positive. If b is an isolated zero of $f - T_{n-1}^*$ then the last sign of $f - T_{n-1}^*$ in $[a, b]$ is negative.*

Proof. Since for every $a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b$ and every $T_{n-1} \in \Lambda_{n-1}$

$$\Delta(f - T_{n-1}, t_0, t_1, ..., t_n) = \Delta(f, t_0, t_1, ..., t_n) \geqslant 0 \tag{2.9}$$

we get $f - T_{n-1}^* \in C(u_0, u_1, ..., u_{n-1})$. Thus, the proof of the theorem follows directly from Theorem 1.1, Lemma 2.1 and Lemma 2.2.

(Obviously, for $f(x)$ satisfying (2.2) the possibility of nonisolated zeroes of $f - T_{n-1}^*$ is excluded).

THEOREM 2.2. *Let $T_n^* = \sum_{i=0}^n a_i u_i$ be any p.b.a. to $f \in C[a, b]$ from Λ_n in a sign-monotone norm $\|\cdot\|$. Then for f satisfying* (2.1) $a_n \geqslant 0$, *while for f satisfying* (2.2) $a_n > 0$.

Proof. Suppose (2.1) is satisfied and assume $a_n < 0$. Then, for every $a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b$

$$\Delta(f - T_n^*, t_0, t_1, ..., t_n) = \Delta(f, t_0, t_1, ..., t_n) - a_n \Delta \begin{pmatrix} u_0, u_1, ..., u_n \\ t_0, t_1, ..., t_n \end{pmatrix} > 0.$$

Therefore $\{u_0, u_1, ..., u_{n-1}, f - T_n^*\}$ is a T-System on $[a, b]$ implying that $f - T_n^*$ has only isolated zeroes in $[a, b]$ with $\tilde{z}(f - T_n^*) < n + 1$, in contradiction to Theorem 1.1.

Suppose now that (2.2) is satisfied, while $a_n = 0$. This means that $T_n^* \in \Lambda_{n-1}$ and by (2.3)

$$\Delta(f - T_n^*, t_0, t_1, ..., t_{n-1}) = \Delta(f, t_0, t_1, ... t_n) > 0$$

for every $a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b$, and we are led to the same contradiction as above.

As is proved in [1] and [3] for the L_p-norms, $1 < p \leqslant \infty$, assumption (2.1) together with the assumption that $f \notin \Lambda_{n-1}[a, b]$ imply $a_n > 0$. This however is not true for every sign-monotone norm, as is demonstrated in [3] by an example concerning best approximation in the L_1-norm.

We conclude this section by proving

THEOREM 2.3. *Let $f \in C[a, b]$ satisfy* (2.2) *and let $E_{n-1}(f) = E_{n-1}(f, \|\cdot\|)$ and $E_n(f) = E_n(f, \|\cdot\|)$ be its degrees of approximation by polynomials from Λ_{n-1} and Λ_n respectively, with respect to a sign-monotone norm $\|\cdot\|$. Then*

$$E_{n-1}(f) > E_n(f).$$

Proof. Since $\Lambda_{n-1} \subset \Lambda_n$ we obviously have $E_{n-1}(f) \geqslant E_n(f)$. If $E_{n-1}(f) = E_n(f)$ then every p.b.a. to f from Λ_{n-1} is also a p.b.a. to f from Λ_n for which $a_n = 0$. This contradicts Theorem 2.2.

3. SUFFICIENT CONDITIONS FOR f TO BE GENERALIZED CONVEX

As is shown in [1], [2] and [3] by a general category argument, there is no direct converse to the results in Section 2 for the L_p-norms $1 \leqslant p \leqslant \infty$. Yet, converse theorems involving conditions on all subintervals of $[a, b]$ are easily derived in case $\{u_0, u_1, ..., u_{n-1}, u_n\}$ is an Extended Complete T-System (ECT-System) [4] and $f \in C^{(n)}[a, b]$.

In this case

$f \in C(u_0, u_1, ..., u_{n-1})$ on $[\alpha, \beta]$

if and only if $D_{n-1}D_{n-2} \cdots D_0 f \geqslant 0$ on $[\alpha, \beta]$ (3.1) where $D_{n-1}, D_{n-2}, \ldots, D_0$ are the differential operators associated with the ECT-System ([4] Chap. XI).

In order to formulate the next results, let $T^*_{n-1}(f, \|\cdot\|_I)$, $T^*_n(f, \|\cdot\|_I)$, $a_n(f, \|\cdot\|_I)$, $E_{n-1}(f, \|\cdot\|_I)$ and $E_n(f, \|\cdot\|_I)$ be the same as above, with respect to the restriction of f to the subinterval $I \subset [a, b]$ and a sign-monotone norm $\|\cdot\|_I$ defined on $C(I)$.

First we prove a converse to Theorem 2.2:

THEOREM 3.1. *Let $\{u_0, u_1, \ldots, u_n\}$ be an ECT-System on $[a, b]$ and $f \in C^{(n)}[a, b]$. If for every subinterval $I \subset [a, b]$ there exists a sign-monotone norm $\|\cdot\|_I$ and an element of best approximation $T_n(f, \|\cdot\|_I)$ to f from Λ_n such that $a_n(f, \|\cdot\|_I) \geqslant 0$, then $f \in C(u_0, u_1, \ldots, u_{n-1})$.*

Proof. In view of (3.1) and continuity arguments, $f \notin C(u_0, u_1, \ldots, u_{n-1})$ implies the existence of a subinterval $I = [\alpha, \beta] \subset [a, b]$ on which $\Delta(-f, t_0, t_1, \ldots, t_n) > 0$ for all $\alpha \leqslant t_0 < t_1 < \cdots < t_n \leqslant \beta$. By Theorem 2.2 this means that $a_n(f, \|\cdot\|_I) < 0$ for every p.b.a $T_n(f, \|\cdot\|_I)$ to f from Λ_n on $I = [\alpha, \beta]$ and every sign monotone norm $\|\cdot\|_I$ defined on $C[\alpha, \beta]$, which is a contradiction.

In the last theorem, there is no connection between the norms which are associated with the subintervals of $[a, b]$. The proofs of the converses to Theorems 2.1 and 2.3, however, rely on the continuous change of the polynomials of best approximation with respect to the change of the interval. Therefore, the next theorem is formulated in terms of a continuous class of sign-monotone norms, and makes use of the following observation:

LEMMA 3.1. *Let $\{\|\cdot\|_I, \|\cdot\|_I$ is defined on $C(I)$, $I = [\alpha, \beta] \subset [a, b]\}$ be a continuous class of sign-monotone norms, and let $f \in C[a, b]$. If $T_n(f, \|\cdot\|_I) = \sum_{k=0}^n a_k u_k$ is a p.b.a to f from Λ_n on I with respect to the sign-monotone norm $\|\cdot\|_I$, then $a_n = a_n(f, \|\cdot\|_I)$ as a multivalued function of the subinterval I, admits all intermediate values.*

Proof. The proof of the lemma follows directly from the definition of a "continuous" class of norms and the fact that for each subinterval $I \subset [a, b]$ the set $\{T \mid T \in \Lambda_n, \|f - T\|_I = E(f, \|\cdot\|_I)\}$ is convex.

THEOREM 3.2. *Let $\{u_0, u_1, \ldots, u_n\}$ be an ECT-System, let $\{\|\cdot\|_I, \|\cdot\|_I$ is defined on $C(I)$, $I = [\alpha, \beta] \subset [a, b]\}$ be a continuous class of sign-monotone norms, and let $f \in C^{(n)}[a, b]$.*

(a) *If for every subinterval $I \subset [a, b]$*

$$E_{n-1}(f, \|\cdot\|_I) > E_n(f, \|\cdot\|_I)$$

then either f or $-f$ is in $C(u_0, u_1, \ldots, u_{n-1})$.

(b) *If for every subinterval* $I \subset [a, b]$ *and every p.b.a* $T_{n-1}^* = T_{n-1}^*(f, \|\cdot\|_I)$, $\tilde{z}(f - T_{n-1}^*) = n$ *then either* f *or* $-f$ *is in* $C(u_0, u_1, \ldots, u_{n-1})$.

Proof. If neither f nor $-f$ is in $C(u_0, u_1, \ldots, u_{n-1})$ then the assumptions on f and on $\{u_0, u_1, \ldots, u_n\}$ together with Lemma 3.1 imply the existence of a subinterval $I \subset [a, b]$ for which $a_n(f, \|\cdot\|_I) = 0$. On this subinterval $E_n(f, \|\cdot\|_I) = E_{n-1}^*(f, \|\cdot\|_I)$. Moreover any p.b.a $T_{n-1} = T_{n-1}^*(f, \|\cdot\|_I)$ to f from Λ_{n-1} on I with respect to $\|\cdot\|_I$ is also a p.b.a to f from Λ_n and thus either $\tilde{z}(f - T_{n-1}^*) > n$ or $f - T_{n-1}^*$ vanishes on a subinterval. Thus, a contradiction is achieved and the theorem is proved.

The question arises whether the last theorems are valid without the assumptions that $\{u_0, u_1, \ldots, u_n\}$ is an ECT-System and $f \in C^{(n)}[a, b]$, as is proved for the sup-norm in [1]. Another question is whether the assumption that the norms in Theorem 3.2 constitute a continuous class—which is essential to the proof of this theorem—can be removed.

For $n = 1$ we give an affirmative answer to these questions, concerning part (b) of Theorem 3.2. In this special case the nature of the problem is quite simple and thus its proof does not seem to have a direct generalization for $n > 1$.

THEOREM 3.3. *Let* $f \in C[a, b]$, *and let* $u_0(x) \in C[a, b]$ *be a positive function. If for every subinterval* $I \subset [a, b]$ *there exists a sign-monotone norm* $\|\cdot\|_I$, *and p.b.a* $T_0^* = T_0^*(f, \|\cdot\|_I) = c_0^* u_0$ *such that* $\tilde{z}(f - T_0^*) = 1$, *then either* f *or* $-f$ *is in* $C(u_0)$.

Proof. If both f and $-f$ are not in $C(u_0)$, then there are points α, β, $a \leqslant \alpha < \beta \leqslant b$, such that

$$\Delta \begin{pmatrix} u_0 & f \\ \alpha & \beta \end{pmatrix} = 0.$$

We show that in $I = [\alpha, \beta]$, $\tilde{z}(f - T_0^*) > 1$ for every p.b.a. T_0^* to f from Λ_0 in every sign-monotone norm. Indeed, by Theorem 1.1, $\tilde{z}(f - T_0^*) \geqslant 1$. Suppose $\tilde{z}(f - T_0^*) = 1$ and denote by $z \in [\alpha, \beta]$ the zero of $f - T_0^*$. Since

$$\Delta \begin{pmatrix} u_0 & f \\ \alpha & \beta \end{pmatrix} = \Delta \begin{pmatrix} u_0 & f - T_0^* \\ \alpha & \beta \end{pmatrix} = \begin{vmatrix} u_0(\alpha) & u_0(\beta) \\ f(\alpha) - T_0^*(\alpha) & f(\beta) - T_0^*(\beta) \end{vmatrix} = 0$$

and since $u_0(\alpha) > 0$, $u_0(\beta) > 0$ then $z = \alpha$ ($z = \beta$) would imply $f(\beta) - T_0^*(\beta) = 0$ ($f(\alpha) - T_0^*(\alpha) = 0$) in contradiction to the assumption $\tilde{z}(f - T_0) = 1$. But if $z \in (\alpha, \beta)$ then

$$\text{sign}(f(\alpha) - T_0^*(\alpha)) = -\text{sign}(f(\beta) - T_0^*(\beta))$$

and the determinant

$$\Delta \begin{pmatrix} u_0 & f \\ x & \beta \end{pmatrix} \neq 0,$$

which is again a contradiction.

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